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Extensions of Endpoint Equivalent and Periodic Tchebycheff Systems

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PRELIMINARIES

Let $n \geqslant 0$ be an integer. Given an ordered set A of cardinality at least $n+2$, a sequence of real functions $\{y_i\}_{i=0}^n$ defined on A will be called a *Tchebycheff system* (T-system) on A, provided that, for every sequence $\{t_0, ..., t_n\}$ of points from A such that $t_0 < \cdots < t_n$, the sign of $\det(y_i(t_j))_{i=0}^n$ is constant and non-zero. For greater flexibility of notation, this determinant will often be written $\det(y_0, ..., y_n/t_0, ..., t_n)$.

The sequence $(y_i\}_{i=0}^n$ will be called a *complete Tchebycheff system* (CT-system) or *Markov system* provided that $\{y_i\}_{i=0}^k$ is a T-system for each $k \in \{0, ..., n\}$. A CT-system (Markov system) is said to be *normed* or *normalized* if y_0 is the constant function 1.

A function f is said to be periodic with period p if, for each point t in its (real) domain, the point $t+kp$ is also in its domain for any integer k, and furthermore $f(t)=f(t+kp)$. We shall say that a sequence of real functions $\{y_0, ..., y_n\}$ is a *periodic T-system* on a set A (real, bounded, and of cardinality $\geqslant n+2$) when the functions are periodic with period equal to the length of the set A and are a T-system on A, the set A containing either its infimum, l_1, or its supremum, l_2.

Similar to periodicity but not totally coincident is the concept of "endpoint equivalence." A real function f defined on a real, nonempty set A will be called *endpoint equivalent* provided that, for all sequences $\{x_n\}$, $\{y_n\}$ in A, such that $x_n \to l_1$ and $y_n \to l_2$, the limits $\lim f(x_n)$, $\lim f(y)$ exist (finite or infinite), and are equal. A T-system defined on such a set A will be called endpoint equivalent if the functions in it are endpoint equivalent. Although the functions considered should be real valued, one can often permit A to be a subset of the extended real number system. The advantage of this will be seen in what follows.

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Of interest are underlying sets which have property (B): The set A is said to have property (B) if, for every two points in A, there is a third point in A between them. Our definition differs from that of Zielke [9] in that he in addition requires $l_1 \notin A$ and $l_2 \notin A$.

Introduction

The purpose of this communication is to present the following results on extension of (or existence of adjoined functions for) endpoint equivalent T-systems:

THEOREM. *Let A be a set with property* (B), *containing either l_1 or l_2, and let $\{x_0, ..., x_n\}$ be an endpoint equivalent T-system on A. Then there exist two functions z_1 and z_2 such that $\{x_0, ..., x_{2n}, z_1, z_2\}$ is also an endpoint equivalent T-system on A. If in addition the functions $x_0, ..., x_n$ are continuous, then z_1 and z_2 are continuous.*

COROLLARY. *Let $\{u_0, ..., u_{2n}\}$ be a T-system of continuous functions on the unit circle K. Then there exist two continuous functions u_{2b+1}, u_{2n+2} such that also the system $\{u_0, ..., u_{2n+2}\}$ is a T-system on K.*

Before giving the proof of the theorem, we digress to provide some perspective:

Antecedents of the Theorem and Corollary. Theorem 18.3 of Zielke [10] states, under the hypothesis that the given T-system is periodic and consists of $2n$ times differentiable functions, that two functions may be adjoined. Earlier, in [4], Zalik claimed that the result stated in our corollary was true. However, the proof was incomplete in that the integral representation used in the proof, announced by Rutman in [1], turned out to be incorrect. This was noted by Zielke (see [2]), and independently by Zalik [7], who gave a more complicated representation under weaker hypotheses. In January 1980, however, Zielke remarked to the second author of this paper that also the representation in [7] is erroneous, and in [11] he gave a counterexample and a correct integral representation. Zielke's results was further extended by Zalik in [8]. Our discussion of periodic and endpoint equivalent T-systems will be based on the recent results of [8, 11].

On Periodicity and Endpoint Equivalence. One of the distinctions between these two concepts is that endpoint equivalence requires neither of the points l_1, l_2 to be finite. Thus, not every endpoint equivalent function is periodic. Another distinction lies in the fact that a function, in order to be periodic, need not be continuous at any point, but, in order to be endpoint equivalent, must be continuous at the endpoints of its domain. Indeed, a function defined, for example, on **R**, and endpoint equivalent on *any* interval of length p, would necessarily be a periodic function with period p, and continuous on **R**. The distinction also has consequences in regard to the stated theorem, perhaps best illustrated by the following examples:

(i) Let us denote by $[t]$ the greatest integer in t. Then the set $\{1, t-[t], (t-[t])^2\}$ is a periodic T-system of period one. If we take as the underlying set the interval $[0, 1)$, the system is not endpoint equivalent. Extension of this system to one containing five functions is very easy: The set $\{1, (t-[t]), (t-[[t])^2, (t-[t])^3, (t-[t])^4\}$ is clearly a periodic T-system of period one, but with all but the first of the functions discontinuous at the integers.

In the general case of any periodic T-system $\{y_0, ..., y_{2n}\}$ on an interval $[a, b)$, we can use known general methods ([3, 8]) to adjoin two functions, say z_1 and z_2, so that the system $\{y_0, ..., y_{2n}, z_1, z_2\}$ is also a T-system on $[a, b)$, and then extend periodically the functions z_1 and z_2 to the whole real line. This procedure, however, does not guarantee that the functions z_1 and z_2, as thus extended, will be continuous at the endpoints of $[a, b)$, even if the functions y_i were continuous there. Indeed, in our next example we adjoin two functions with discontinuities to a periodic T-system of continuous functions, obtaining a larger periodic T-system.

(ii) Consider the set $\{1, \cos t, \sin t\}$, a periodic T-system on the interval $[0, 2\pi)$. Integrating twice, we conclude that $\{1, t, t^2, \cos t, \sin t\}$ is a T-system on $[0, 2\pi)$, and therefore also $\{1, \cos t, \sin t, t, t^2\}$ is a T-system there. The functions may now be extended by periodicity, in the manner used in example (i). Lemma 2 of this paper (see below) provides the justification for the assertion that the augmented system is also a T-system on the original interval.

In our final example, we begin with a periodic T-system and obtain an extension containing an unbounded function, leaving no possibility of obtaining a periodic extension in the usual sense.

(iii) We begin with the periodic T-system $\{1, \cos 2t, \sin 2t\}$, defined on the interval $[-\pi/2, \pi/2)$, observing that another basis for its span is the set $S = \{\cos^2 t, \sin t \cos t, \sin^2 t\}$. The three functions in S may be obtained by multiplying respectively the functions in $\{1, \tan t, \tan^2 t\}$, a T-system on the *open* interval $(-\pi/2, \pi/2)$, by $\cos^2 t$. Thus, the unbounded function $\tan^3 t \cos^2 t$ can be adjoined to S, obtaining a set which is clearly a T-system on the open interval $(-\pi/2, \pi/2)$.

We thus conclude that the significant problem in extension of periodic T-systems is that of preserving continuity at the endpoints, which is effected by adopting the concept of endpoint equivalence. Approach to the problem of extensions via endpoint equivalent functions has another advanage, in that the problem is equally as meaningful on infinite or semi-finite intervals, as it is on finite intervals in the case of periodic functions.

Existing Results

We list here the existing results upon which the proof of our theorem is based.

To facilitate the statement of our first necessary result, we adopt the term *strongly representable* to describe a sequence of functions $\{y_0, ..., y_n\}$ $(n \geqslant 1)$ if, given any point c in A, there exist functions $u_0, ..., u_n$ such that, for each $k \in \{0, ..., n\}$, the sequence $\{u_0, ..., u_k\}$ is a basis for the span of $\{y_0, ..., y_k\}$; a strictly increasing real function h defined on A satisfying $h(c) = c$; and continuous increasing real functions $w_1, ..., w_n$ defined on $(h(l_1^+), h(l_2^-))$ and strictly increasing on $h(A) \cap (h(l_1^+), h(l_2^-))$ such that, for all x in $A \backslash \{l_1, l_2\}$

$$\begin{aligned} u_0 &= 1 \\ u_1(x) &= \int_c^{h(x)} dw_1(t_1) \\ &\vdots \\ u_n(x) &= \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_1(t_1). \end{aligned} \tag{1}$$

The functions $u_0, ..., u_n$ defined above will be called an *integral representation*. We remark that the values for $u_0, ..., u_n$ which might exist at l_1 or l_2 are not necessarily those which would be obtained from the equations in (1).

We will say that the functions $\tilde{u}_0, ..., \tilde{u}_n$ possess a *canonical integral representation* if the function h in (1) is the identity function. If $u_0, ..., u_n$ are as in (1), we may make the correspondence $\tilde{u}_i = u_i^0 h^{-1}$ on $h(A)$, writing for $x \in (h(l_1^+), h(l_2^-))$

$$\begin{aligned} \tilde{u}_0 &= 1 \\ \tilde{u}_1(x) &= \int_c^{x} dw_1(t_1) \\ &\vdots \\ \tilde{u}_n(x) &= \int_c^{x} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_1(t_1). \end{aligned} \tag{2}$$

In such a case, we say that the functions described in (2) are a canonical (integral) representation for the functions $u_0, ..., u_n$ of (1). We note that the functions defined in (2) will be a normalized Markov system on $h(A)$ if and only if the original functions are a normalized Markov system on A. We have

THEOREM A. *Let A have property* (B) *and contain neither l_1 nor l_2. Then $\{y_0, ..., y_n\}$ $(n \geqslant 1)$ is a normalized Markov system if and only if it is strongly representable.*

The linear span of a Tchebycheff system is called a *Haar space.* With this definition, we have:

THEOREM B [10, Theorem 7.7]. *Let $M \subseteq \mathbf{R}$ be a set which contains neither its infimum nor its supremum, and U an n-dimensional Haar space, $n \geqq 1$. Then U contains an $(n-1)$-dimensional Haar space.*

Remarks. It is readily seen that if $\{y_0, ..., y_n\}$ is strongly representable, it is a normalized Markov system. The converse has been proved by Zielke [10, Corollary 3′]. A generalization of Theorem A has recently been obtained by Zalik [8]. Theorem B is due to Zalik [6], and an earlier version, also sufficient for present purposes, appeared in Zielke [9].

LEMMAS

We now state a series of lemmas whose combined effect is to complete the proof of our Theorem, as will be described in the final section of this paper. The first lemma, a consequence of Theorem A, is analogous to a result which is well known for T-systems defined on an open interval. This older result will also be found useful here and will appear as Lemma 2.

LEMMA 1. *Let A be any set with property* (B) *containing neither its infimum l_1 nor its supremum l_2, and let $\{u_0, ..., u_n\}$ $(n \geqslant 1)$ be a normalized Markov system defined on A, with its integral representation* (1) *and a canonical representation* (2) *(as guaranteed by Theorem* A*). Let w be a continuous increasing function defined on $(h(l_1^+), h(l_2^-))$ which is strictly increasing on $h(A)$. Then*

(a) *Let $v_0 = 1$, and, for $i \in \{1, ..., n+1\}$, let*

$$v_i(x) = \int_c^{h(x)} \tilde{u}_{i-1}(t)\, dw(t).$$

Then the set $\{v_0, ..., v_{n+1}\}$ is a normalized Markov system on A.

(b) *If moreover* v_1 *is continuous on* A, *then also the functions* $v_2, ..., v_{n+1}$ *are continuous on* A.

Proof. Integration of each of the functions exhibited in (2) with respect to the function w must result in a set $\{\tilde{v}_0, ..., \tilde{v}_{n+1}\}$ of functions defined on the set $h(A)$ by $\tilde{v}_0(t) = 1$ and

$$\tilde{v}_i(t) = \int_c^x \tilde{u}_{i-1}(x)\, dw(x), \qquad \text{for} \quad i \in \{1, ..., n+1\}.$$

We now define $v_i = \tilde{v}_i \circ h$ for $i \in \{0, ..., n+1\}$, and, noting that we have a representation of form (1), we invoke Theorem A to show that we have a normalized Markov system.

To see that continuity of v_1 implies continuity of $v_2, ..., v_{n+1}$, let b be any point in A such that $b \geqq c$, and let x and y be points of A such that $c \leqq x \leqq y \leqq b$. Let $M_i = \tilde{u}_{i-1}(h(b))$ for $i \in \{2, ..., n+1\}$. We immediately have

$$\begin{aligned} 0 \leqq v_i(y) - v_i(x) &= \int_{h(x)}^{h(y)} \tilde{u}_{i-1}(t)\, dw(t) \leqq \int_{h(x)}^{h(y)} M_i dw(t) \\ &= M_i(v_1(y) - v_1(x)). \end{aligned}$$

Since v_1 is assumed to be continuous, the continuity of v_i follows for any point in A which lies in (c, b). Since b is arbitrary, the continuity follows for any point of A which lies to the right of c. Continuity at the other points of A may be shown in similar fashion.

Remark. Part (b) of the above lemma is similar to [5, Lemma 2].

The following lemma records a well-known result which will form a component of the proof of Lemma 3. The fact that the underlying set is an interval allows one to bypass the techniques used in the previous lemma. Also, it is not in an obvious sense implied by Lemma 1; the integration takes place on A itself, not on $h(A)$. The reader may consult, for example, [10, Lemma 13.2], where the statement is proved under weaker hypotheses, but the underlying set is an open interval.

LEMMA 2. *Let* $I(a, b)$ *denote any interval with endpoints* a *and* b. *Let* U *be an* n*-dimensional Haar space of continuous functions defined on* $I(a, b)$, *and let* g *be a continuous, strictly monotone function on* $I(a, b)$. *Let* c *be a point in* $I(a, b)$. *Then the space*

$$V = \{h \mid h(x) = \int_c^x f(t)\, dg(t) + C, f \in U, C \in \mathbf{R}, x \in I(a, b)\}$$

is an $(n+1)$*-dimensional Haar space on* $I(a, b)$.

Proof. The proof of this statement follows by straightforward application of the Mean Value Theorem for Riemann–Stieltjes integrals.

The following lemma generalizes [10, Lemma 18.1]; the set A is not required to be an open interval here. With the incorporation of small refinements, the proof used here is essentially that of [10]. The proof of the corresponding lemma in [4] employed a construction of the functions u and v which did not yield the required boundedness.

LEMMA 3. *Let $A \subseteq \mathbf{R}$ and let $\{1, f\}$ be a normalized Markov system on A. Then there exist functions u and v, bounded and continuous on* **R** *such that $\{1, u \circ f, v \circ f, f\}$ is a normalized Markov system on A.*

Proof.
Let

$$u(x) = \int_0^x e^{-t^2}\, dt, \qquad \text{and} \qquad v(x) = e^{-x^2}.$$

Invoking Lemma 2, we note that $\{1, u, v, x\}$ will be a Markov system if $\{u', v', 1\}$ is a Markov system, and the latter will be a Markov system if $\{1, v'/u', 1/u'\}$ is, which will in turn be a Markov system if $\{(v'/u')', (1/u')'\}$ is. The result follows immediately because

$$v'(x)/u'(x) = -2x, \qquad \text{and} \qquad 1/u'(x) = e^{x^2}.$$

Now, since $\{1, u, v, x)$ is a normalized Markov system on **R**, it is a fortiori a normalized Markov system on $f(A)$, and the conclusion readily follows.

LEMMA 4. *Let the set A have property* (B) *and contain neither its infimum l_1 nor its supremum l_2. Let $n > 1$, and let $\{y_0, ..., y_n\}$ be a set of endpoint equivalent functions defined on $A \cup \{l_1, l_2\}$, having a representation on A of the form*

$$y_i(t) = y_0(t)\, u_i(t), \qquad i = 0, 1, ..., n\ (\geqslant 1), \tag{3}$$

where the functions $u_0, ..., u_n$ form an integral representation as defined in (1), *and y_0 is strictly positive on A. If $y_i(l_1) = y_i(l_2) = 0$ for $i \in \{0, ..., n-1\}$, then there exist two functions z_1 and z_2 defined on $A \cup \{l_1, l_2\}$, such that $\{y_0, ..., y_{n-1}, z_1, z_2, y_n\}$ is a Markov system on A, and $z_i(l_1) = z_i(l_2) = 0$, for $i \in \{1, 2\}$. If $y_0, ..., y_n$ are continuous, then also z_1 and z_2 are continuous.*

Proof. Consider the function $f(t) = \int_c^t dw_n(s)$, where c and w_n are as in (1). Since w_n is strictly increasing on $h(A)$, we note that $\{1, f\}$ is a normalized Markov system on $h(A)$. We thus conclude, using Lemma 3, that there exist two functions u and v, continuous and bounded on **R**, such that the set $\{1, u\circ f, v\circ f, f)$ is a normalized Markov system on $h(A)$. Invoking Lemma 1, we may perform repeated integrations on the functions in this Markov system with respect to the weight functions $w_{n-1}, ..., w_1$ obtaining at length a normalized Markov system $\{\tilde{u}_0, ..., \tilde{u}_{n-1}, q_1, q_2, \tilde{u}_n\}$, where $u_0, ..., \tilde{u}_b$ are a canonical representation of form (2) for the original functions $u_0, ..., u_n$ defined as in (1) which are part of our hypotheses. The functions q_1 and q_2 are defined by

$$q_1(t)=\int_c^t\int_c^{s_1}\cdots\int_c^{s_{n-2}}(u\circ f)(s_{n-1})\,dw_{n-1}(s_{n-1})\cdots dw_1(s_1),$$

and

$$q_2(t)=\int_c^t\int_c^{s_1}\cdots\int_c^{s_{n-2}}(v\circ f)(s_{n-1})\,dw_{n-1}(s_{n-1})\cdots dw_1(s_1),$$

where c and $w_1, ..., w_{n-1}$ are as in (1) or (2). Defining for $i\in\{1, 2\}$ $z_i(t)=y_0(t)(q_i\circ h)(t)$, and with the definition of the functions $y_0, ..., y_n$ as aforementioned, the set $\{y_0, ..., y_{n-1}, z_1, z_2, y_n\}$ is a Markov system on A. The vanishing of the functions z_1 and z_2 at the points l_1 and l_2 follows from the boundedness of u and v and from the hypothesis that y_0 is zero at those points. To see this, let M be a bound for $|u|$ and $|v|$. Then, for any $t\in A$, and for $i=1$ or $i=2$ we have

$$|z_i(t)|=|y_0(t)|\;|q_i(t)|\leqq M\;|y_0(t)|\;|u_{n-1}(t)|.$$

The right side of this inequality is endpoint equivalent and has the value zero at l_1 and l_2 by hypothesis.

If the functions $y_0, ..., y_n$ were continuous, we readily infer the continuity of $u_1, ..., u_n$, and in particular of u_1, and the continuity of z_1 and z_2 would then follow by (b) of Lemma 1.

The following lemma generalizes a well-known result about periodic continuous functions defined on a closed interval (cf., e.g., [4, Lemma 4]). We note that endpoint equivalence is the weakest hypothesis upon which the proof can be based.

LEMMA 5. *Let $\{y_0, ..., y_{2n-1}\}$ be a set of endpoint equivalent functions defined on a set $A\cup\{l_1, l_2)$ which constitutes a T-system on $A\backslash\{l_1, l_2\}$. Then for any selection of points $t_0, ..., t_{2n-2}$ in A,* $\det(y_0, ..., y_{2n-1}/t_0, ..., t_{2n-2}, l_2))=0$.

Proof. Without loss of generality, we may assume that the points $l_1, t_0, ..., t_{2n-2}, l_2$ are strictly ordered from left to right, and that the sign of the determinant which occurs in the definition of a T-system is positive. Let

$\{x_k\}$ be any sequence of points in A converging to l_1 and satisfying $x_k < t_0$ for all k. Let $\{y_m)$ be any sequence of points from A converging to l_2 and satisfying $y_m > t_{2n-2}$ for all m. We have

$$0 < \det(y_0, ..., y_{2n-1}/x_k, t_0, ..., t_{2n-2}) \qquad \text{for all } k,$$

and

$$0 < \det(y_0, ..., y_{2n-1}/t_0, ..., t_{2n-2}, y_m) \qquad \text{for all } m.$$

Since the T-system $\{y_0, ..., y_{2n-1}\}$ is endpoint equivalent, we have, taking limits as $x_k \to l_1$ and $y_m \to l_2$,

$$\begin{aligned} 0 &\leqq \det(y_0, ..., y_{2n-1}/l_1, t_0, ..., t_{2n-2}) \\ &= \det(y_0, ..., y_{2n-1}/l_2, t_0, ..., t_{2n-2}) \\ &= -\det(y_0, ..., y_{2n-1}/t_0, ..., t_{2n-2}, l_2) \leqq 0, \end{aligned}$$

and the result follows.

Our final lemma is

LEMMA 6. *Let $\{y_0, ..., y_{2n}\}$ be a set of functions defined on a set $A \cup \{l_1, l_2\}$ which constitutes an endpoint equivalent T-system on $A \cup \{l_2\}$, and assume that $\{y_0, ..., y_{2n-1}\}$ is a T-system on $A \backslash \{l_1, l_2\}$. Then $y_i(l_2) = 0$ for $i \in \{0, ..., 2n-1\}$.*

Proof. We note first of all that certainly $y_{2n}(l_2) \neq 0$. For otherwise, for any points $t_0, ..., t_{2n-1}$ in A, we would have

$$\det(y_0, ..., y_{2n}/t_0, ..., t_{2n-1}, l_2) \\ = \sum_{j=0}^{2n-1} (-1)^j y_{2n}(t_j) \det(y_0, ..., y_{2n-1}/t_0, ..., t_{j-1}, t_{j+1}, ..., t_{2n-1}, l_2),$$

and the expansion of the determinant would consist of a sum of terms which, by Lemma 5, are all zero, and the set $\{y_0, ..., y_n\}$ would not be a T-system on $A \cup \{l_2\}$.

If now, for some $j \in \{0, ..., 2n-1\}$ we have $y_j(l_2) \neq 0$, let $y'_{2n} = y_{2n} + cy_j$ with c so chosen that $y'_{2n}(l_2) = 0$. Clearly $\{y_0, ..., y_{2n-1}, y'_{2n}\}$ is a T-system on A. But from the preceding argument we know that $y'_{2n}(l_2) \neq 0$, and we have obtained a contradiction. The result follows.

PROOF OF THE THEOREM

Let us assume, without loss of generality, that A contains l_2, and that the sign of the determinant that occurs in the definition of a T-system is positive.

If $n=0$, let f be any order-preserving homeomorphism from the interval $(l_1, l_2]$ to the interval $(0, 2\pi]$. Then $x_0(t)\cos f(t)$ and $x_0(t)\sin f(t)$ will serve as two functions which may be adjoined. We assume henceforth that $n>0$.

As a first step, we note that the span of $\{x_0, ..., x_{2n}\}$ is a fortiori a Haar space on the set $A\backslash\{l_1, l_2\}$. By repeated application of Theorem B, we obtain a basis $\{\hat{y}_0, ..., \hat{y}_{2n}\}$ for this space which is a Markov system of endpoint equivalent functions on $A\backslash\{l_1, l_2\}$; these functions are of necessity also defined on $A\cup\{l_1, l_2\}$ and constitute a T-system on A. By Lemma 6, this Markov system must satisfy $\hat{y}_i(l_1)=\hat{y}_i(l_2)=0$ for $i\in\{0, ..., 2n-1)$ and $\hat{y}_{2n}(l_2)>0$. It follows that $\{1, \hat{y}_1/\hat{y}_0, ..., \hat{y}_{2n}/\hat{y}_0\}$ is a normalized Markov system on the set $A\backslash\{l_1, l_2\}$. By Theorem A, this system has a basis having a representation of form (1). Hence, the system $\{\hat{y}_0, ..., \hat{y}_{2n}\}$ has a basis $\{y_0, ..., y_{2n}\}$ having a representation of form (3).

Since the span of $\{x_0, ..., x_{2n}\}$ admits of a basis $\{y_0, ..., y_{2n}\}$ with a representation of form (3), Lemma 4 guarantees the existence of two functions z_1 and z_2 which can be adjoined, in such a manner that z_1 and z_2 are zero at l_1 and l_2, and the augmented set $\{y_0, ..., y_{2n-1}, z_1, z_2, y_{2n}\}$ is a Markov system on $A\backslash\{l_1, l_2\}$. We now use the fact that all of these functions are zero at l_2, except for y_{2n} which has been shown to be positive, to demonstrate that we in fact have a Markov system on the entire set A, including l_2. Assuming that $l_1, t_0, ..., t_{2n+1}, l_2$ are points strictly ordered from left to right, we have

$$\begin{aligned}&\det(y_0, ..., y_{2n-1}, z_1, z_2, y_{2n}/t_0, ..., t_{2n+1}, l_2)\\&\qquad=y_{2n}(l_2)\cdot\det(y_0, ..., y_{2n-1}, z_1, z_2/t_0, ..., t_{2n+1})>0.\end{aligned}$$

We have therefore shown that $\{y_0, ..., y_{2n}, z_1, z_2\}$, is a T-system on A. It follows immediately that also $\{x_0, ..., x_{2n}, z_1, z_2\}$ is a T-system on A.

This completes the proof of the theorem.

References

1. M. A. Rutman, Integral representation of functions forming a Markov series, *Dokl. Akad. Nauk. SSSR* **164** (1965), 989–992.
2. L. L. Schumaker, On Tchebycheffian spline functions, *J. Approx. Theory* **18** (1976), 278–303.
3. R. A. Zalik, Existence of Tchebycheff extensions, *J. Math. Anal. Appl.* **51** (1975), 68–75.
4. R. A. Zalik, Extension of periodic Tchebycheff systems, *J. Math. Anal. Appl.* **56** (1976), 373–378.
5. R. A. Zalik, Smoothness properties of generalized convex functions, *Proc. Amer. Math. Soc.* **56** (1976), 118–120.
6. R. A. Zalik, On transforming a Tchebycheff system into a complete Tchebycheff system, *J. Approx. Theory* **20** (1977), 220–222.

7. R. A. ZALIK, Integral representation of Tchebycheff systems, *Pacific J. Math.* **68** (1977), 553–568.
8. R. A. ZALIK, Integral represenation and embedding of weak Markov systems, *J. Approx. Theory* **58** (1989), 1–11.
9. R. ZIELKE, On transforming a Tchebyshev-system into a Markov-system, *J. Approx. Theory* **9** (1973), 357–366.
10. R. ZIELKE, Discontinuous Čebyšev Systems, "Lecture Notes in Mathematics," Vol. 707, Springer-Verlag, New York, 1979.
11. R. ZIELKE, Relative differentiability and integral representation of a class of weak Markov systems, *J. Approx. Theory* **44** (1985), 30–42.